GEODESICS ON WEIGHTED PROJECTIVE SPACES

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ABSTRACT. We study the inverse spectral problem for weighted projective spaces using wave-trace methods. We show that in many cases one can "hear" the weights of a weighted projective space.

CONTENTS

1. INTRODUCTION

A weighted complex projective space is a quotient

$$X = S^{2d-1}/S^1,$$

where $S^{2d-1} \subset \mathbb{C}^d$ is the unit sphere in $\mathbb{C}^d$ and the circle acts on the sphere as

$$e^{it} \cdot (z_1, \ldots, z_d) = (e^{iN_1 t} z_1, \ldots, e^{iN_d t} z_d),$$

where $N_1, \ldots N_d$ are positive integers, called the weights. Although naturally a (singular) complex manifold, we will think of X as a real Riemannian orbifold, of real dimension $2(d-1)$.

In [ADFG], Abreu, Dryden, Freitas and Godinho raise the interesting question of whether the spectrum of the Laplace-Beltrami operator on X determines the weights $N_1, \ldots, N_d$, and they show that this is indeed the case for $d = 3$, if one knows the spectrum of the Laplacian on zero and 1-forms. Their methods center on the heat invariants and use equivariant cohomology.

In a different direction, the wave-trace formula of [DG] can be extended to compact Riemannian orbifolds, see §2 and [SU]. On the basis of this result one can expect that, generically, the spectrum of the Laplacian on an orbifold determines the set of lengths (with multiplicities) of closed geodesics, together with the dimension of the manifold of closed geodesics of a given length. This is actually the case for weighted projective spaces. Let's call the multiset of such pairs of data, (lengths of closed geodesics with multiplicities, and the dimension of their set), the weighted length spectrum. From the wave-trace point of view, the inverse spectral problem of the previous paragraph is closely related to the problem: *Does the weighted length spectrum of X determine the set of weights $\{N_j\}$?*

Geodesics on X are projections of geodesics on S^{2d-1} that are "horizontal", meaning orthogonal to S^1 orbits. Therefore all geodesics on X are periodic with a common period, 2π. However, there can exist exceptional, shorter geodesics (they are "exceptional" in the sense that the measure of their set in the space of all geodesics is zero). They arise because of the non-trivial isotropy of some points on the sphere S^{2d-1}.

In this paper we show that, in many cases, the weighted length spectrum of X corresponding to the exceptional geodesics determines the set of sums of pairs of weights, $\{N_j + N_k, j < k\}$, with multiplicities. This in turn raises the following problem: *Is a set of d positive numbers, $\{N_j\}$, determined by the multiset of the sums of the pairs $\{N_j + N_k, j < k\}$?* The answer is obviously "no" for $d = 2$ and obviously "yes" for $d = 3$. It has been known for some time that the answer is "yes" if d is not a power of two. For convenience we present a proof of this result in an appendix, since the only reference we found, [SGK], may not be readily available to the reader. Conversely, Elizabeth Chen and Jeffrey Lagarias have shown recently that, if d is any power of two, then there exists sets of d numbers that are not determined by the multiset of the sums of pairs. (personal communication).

To briefly summarize our results, our calculations of the weighted length spectrum of X, together with the wave-trace formula for orbifolds, constitue a proof that for d not a power of two, generically (with respect to the weights) one can "hear" the weights of a weighted projective space.

About the organization of the paper: After reviewing the wave trace for Riemannian orbifolds in §2, we compute in §3 the lengths of the exceptional geodesics in the case $d = 2$, which we feel makes clearer the general discussion in §§4 and 5. Our main inverse spectral result is Theorem 5.2.

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2. THE WAVE TRACE FOR ORBIFOLDS

In this section we outline a proof of the "wave trace formula" for orbifolds, referring to [SU] for details. We recall that an orbifold can always be represented as the quotient of a manifold by a group, i.e., as a quotient X/G where X is a manifold, G a compact connected Lie group and $\tau : G \times X \to X$ a locally free action of G on X. If in addition X is equipped with a G-invariant Riemannian metric the Laplace operator $\Delta_X : C^\infty(X) \to C^\infty(X)$ maps $C^\infty(X)^G$ into itself and via the identification, $C^\infty(X)^G = C^\infty(X/G)$, defines a Laplace operator, $\Delta_{X/G}$ on X/G. Assuming that X is compact the wave trace formula for X asserts that the expression

$$e(t) =: \operatorname{trace} \exp it\sqrt{\Delta_X}$$

is well-defined as a tempered distribution and that its singular support is contained in (and modulo some slightly technical clean intersection assumptions which we'll avoid going into at this juncture) coincides with the period spectrum of X.[*] We will prove below that the analogue of this result is also true for X/G providing one defines carefully what one means by a "closed geodesic" and by its "period".

Let $T^\# X$ be the punctured cotangent bundle of X. From τ one gets an action, $\tau^\#$, of G on $\mathbb{T}^\# X$ and from the symbol of Δ_X a Hamiltonian vector field, v_H, where $H = \sigma(\Delta_X)^{\frac{1}{2}}$. Moreover $\tau^\#$ and the flow generated by v_H commute, so one has an action,

$$(g, t) \to \tau_g^\# \exp tv_H$$

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[*]A point, $T \in \mathbb{R}$ is in the *period spectrum* of X if there exists a closed geodesic, γ, on X of period T, i.e., $\gamma(t + T) = \gamma(t)$ for all t.

of $G \times \mathbb{R}$ on $T^\# X$. The *moment Lagrangian* for this action (see [We])
is the set of points

$$(x, \xi, y, \eta, t, \tau, g, \gamma) \in T^\#(X \times X \times \mathbb{R} \times G)$$

satisfying

$$(2.1) \qquad (y, \eta) = (\tau_g^\# \exp t v_H)(x, \xi), \quad \tau = H(x, \xi), \quad \gamma = \Phi(x, \xi)$$

where $\Phi : T^\# X \to \mathfrak{g}^*$ is the moment map associated with the action
of G on $T^\# X$. The set (2.1) is also the microsupport of the opera-
tor, $\tau_g^\# \exp it\sqrt{\Delta_X}$, and we'll use this fact and functorial properties of
microsupports to compute the microsupport of the distribution

$$(2.2) \qquad \text{trace} \exp it \sqrt{\Delta_{X/G}}$$

Some observations:

1. Modulo the identification, $C^\infty(X)^G = C^\infty(X/G)$ the expression
   (2.2) can be written as an integral

$$(2.3) \qquad \int_G \text{trace}(\tau_g^\# \exp it \sqrt{\Delta_X}) \, dg$$

   where dg is Haar measure.
2. Let $e(x, y, t, g)$ be the Schwartz kernel of the operator, $\tau_g^\# \exp it\sqrt{\Delta_X}$.
   Then (2.3) can also be written as a double integral

$$(2.4) \qquad \int e(x, x, t, g) \, dg \, dx$$

   where dx is the Riemannian volume form on X.
3. Let $\iota_\Delta : X \times \mathbb{R} \times G \to X \times X \times \mathbb{R} \times G$ be the diagonal embedding
   and $\pi : X \times \mathbb{R} \times G \to \mathbb{R}$ the projection, $(x, t, g) \to t$. Then (2.4)
   can be written more functorially as a "pull-back" followed by a
   "push-forward":

$$(2.5) \qquad \pi_* \iota_\Delta^* e(t).$$

4. Functorial properties of "pull-backs" tell us that if $(x, \zeta, t, \tau, g, \gamma)$
   is in the microsupport of $\iota_\Delta^* e$ there exists a point $(x, \xi, y, \eta, t, \tau g, \gamma)$
   in the microsupport of e, i.e., in the set (2.1), such that

$$(2.6) \qquad x = y \text{ and } \zeta = \xi - \eta$$

   and, by (2.1),

$$(2.7) \qquad (x, \eta) = (\tau_g^\# \exp t v_H)(x, \xi).$$

5. Functorial properties of "push-forwards" tell us that if (T, τ) is
   in the microsupport of $\pi_* \iota_\Delta^* e$ there exists a point, $(x, \zeta, T, \tau, g, \gamma)$
   in the microsupport of $\iota_\Delta^* e$, i.e., in the set (2.6)–(2.7) such that
   $\zeta = \gamma = 0$. Thus by (2.6)–(2.7) we've proved:

Theorem. *If T is in the singular support of the distribution, (2.5) there exists an $(x, \xi, g) \in (T^\# X) \times G$ with*

$$(2.8) \qquad (T_g^\# \exp T v_H)(x, \xi) = (x, \xi)$$

and

$$(2.9) \qquad \Phi(x, \xi) = 0 \,.$$

Moreover, modulo appropriate clean intersection assumptions, one can compute the leading term of the singularity of (2.5) at T in terms of the geometry of the manifold of curves $(T_g^\# \exp t v_H)(x, \xi)$.

It remains to show that (2.8) and (2.9) can be interpreted as saying that T is in the period spectrum of X/G. Since the action of $G \times \mathbb{R}$ on $T^\# X$ is locally free the group of (g, T)'s satisfying (2.8) is a discrete group. Thus the projection, $G \times \mathbb{R} \to \mathbb{R}$ maps this onto a discrete subgroup of $\mathbb{R}$ and hence onto a lattice group

$$(2.10) \qquad \{nT_0 \,, \, n \in \mathbb{Z}\} \,.$$

Moreover since G is compact the projection of this group onto G is a finite subgroup of order m. Thus by (2.8)

$$(\exp mT v_H)(x, \xi) = (x, \xi)$$

showing that the curve

$$(\exp t v_H)(x, \xi) \,, \, -\infty < t < \infty$$

is periodic of period mT and hence its projection onto X is a closed geodesic, $\gamma(t)$, of period mT.

Let's next unravel the meaning of (2.9). This asserts that the covector, $\xi \in T_x^\#$, is conormal to the orbit of G through x and hence for the projected curve, $\gamma(t)$, that $\frac{d\gamma}{dt}(0)$ is orthogonal at x to the orbit of G through x. Since $\Phi^{-1}(0)$ is $G \times \mathbb{R}$ invariant, this holds as well at $\gamma(t)$, therefore the implication of (2.9) is that at every point, $\gamma(t)$, $-\infty < t < \infty$, the orbit of G through $\gamma(t)$ is orthogonal to $\gamma(t)$. In other words, γ is horizontal with respect to the quasi-fibration, $\mathrm{pr} : X \to X/G$, and therefore its projection, $\mathrm{pr} \circ \gamma$, onto X/G is a geodesic on X/G.[†] Finally note that if T belongs to the set (2.10) then

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[†]Away from singular points these projections are locally length-minimizing, and therefore worthy of the name "geodesics" on X/G. Across singular strata they fail to be locally lenght-minimizing, but we will continue to call them geodesics, as no locally length minimizing curves exist in that case. Of course one has to check that this definition is independent of the presentation of X/G as the quotient of X by G, but this isn't hard to do.

by (2.8) there exists a $g \in G$ such that

$$(\tau_g^\# \exp T v_H)(x, \xi) = (x, \xi)$$

and hence

$$\gamma(0) = \tau_g \gamma(T)\,.$$

Thus for the projection of γ onto X/G

$$\mathrm{pr} \circ \gamma(T) = \mathrm{pr} \circ \gamma(0)\,.$$

In other words, T is the period of the closed geodesic, $\mathrm{pr} \circ \gamma(t)$, $0 \le t \le T$. In addition, the calculation alluded to in the previous theorem shows that in positive curvature there cannot be singularity cancellations from geodesics of a given length. Therefore one has:

Theorem. *If $T \in \mathbb{R}$ is in the singular support of the distribution (2.2), there exists a closed geodesic on X/G of period T. Moreover, if appropriate clean intersection hypotheses are satisfied and X has positive sectional curvature, this result is an "if and only if" result.*

3. Weighted projective lines

These spaces are obtained as quotients of the three sphere $S^3 \subseteq \mathbb{C}^2$ by circle actions. More explicitly, let p and q be positive integers with $1 < p < q$ and g.c.d.$(p, q) = 1$, and let

$$\tau : S^1 \times S^3 \to S^3$$

be the action $\tau(e^{i\theta})(z_1, z_2) = (e^{ip}z_1, e^{iq}z_2)$. Then $X = S^3/S^1$ is an orbifold with singularities at its north pole $[1, 0]$ and south pole $[0, 1]$. Moreover we can define a Riemannian metric on the non-singular part of X, as follows. Let v be the infinitesimal generator of the action τ. Since this action is locally free $v(p) \ne 0$ at all $p \in S^3$. Let π be the projection of S^3 onto the quotient X. Then, except at the north and south poles, we can define the metric on $T_q X$, $q = \pi(p)$, by identifying $T_q X$ with the horizontal part H_p of $T_p S^3$, i.e. with the space in T_p orthogonal to v_p. Since the S^1 action is by isometrics this definition doesn't depend on p. Moreover if $\gamma : [0, 2\pi] \to S^3$ is a geodesic which is horizontal at $p = \gamma(0)$, i.e. satisfies

$$\left\langle \frac{d\gamma}{dt}(0)\,, v_p \right\rangle = 0,$$

then by parallel transport it's horizontal at $q = \gamma(t)$ so it makes sense to talk about *horizontal* geodesics in S^3, and it's clear from our definition of the Riemann metric on X that their projections are geodesics on X. In particular X is Zoll: all its geodesics are closed.

3.1. **Horizontal geodesics.** Here's a more down-to-earth description of these horizontal geodesics. Every geodesic on S^3 is the intersection of S^3 with a two-dimensional subspace, V, of $\mathbb{R}^4$. Identify $\mathbb{R}^4$ with $\mathbb{R}^2 \oplus \mathbb{R}^2$ and let $J : \mathbb{R}^2 \to \mathbb{R}^2$ be the map $J(a, b) = (-b, a)$. (In other words, via the identification

$$\mathbb{R}^2 \to \mathbb{C} \qquad (x, y) \to x + \sqrt{-1}\, y \,,$$

let J be multiplication by $\sqrt{-1}$.) Then the vector field $v(\mathrm{v})$ at $\mathrm{v} = (\mathrm{v}_1, \mathrm{v}_2) \in \mathbb{R}^2 \oplus \mathbb{R}^2$ is just

(3.1) $$\tilde{J}\mathrm{v} = (pJ\mathrm{v}_1, qJ\mathrm{v}_2) \,.$$

Thus if V is a two-dimensional subspace of $\mathbb{R}^4$ spanned by v and w with $|\mathrm{v}| = |w| = 1$ and $\mathrm{v} \perp w$, and $\gamma : [0, 2\pi] \to S^3$ is the parametrized geodesic lying on the circle $V \cap S^3$ with $\gamma(0) = \mathrm{v}$ and $\dfrac{d\gamma}{dt}(0) = w$,

$$\left\langle v(\mathrm{v}), \frac{d\gamma}{dt}(0) \right\rangle = 0 \Leftrightarrow \left\langle \tilde{J}\mathrm{v}, w \right\rangle = 0 \,.$$

We conclude:

Theorem 3.1. *Let $\tilde{\omega} \in \Lambda^2(\mathbb{R}^4)^*$ be the two-form*

(3.2) $$\tilde{\omega}(\mathrm{v}, w) = \langle \tilde{J}\mathrm{v}, w \rangle$$

and let $V \subseteq \mathbb{R}^4$ be a two-dimensional subspace of $\mathbb{R}^4$. Then the geodesic $V \cap S^3$ is horizontal iff V is Lagrangian with respect to the two-form $\tilde{\omega}$.

Note that if $\mathrm{v} = (\mathrm{v}_1, \mathrm{v}_2) \in \mathbb{R}^2 \oplus \mathbb{R}^2$ and $w = (w_1, w_2) \in \mathbb{R}^2 \oplus \mathbb{R}^2$ then

(3.3) $$\tilde{\omega}(\mathrm{v}, w) = p\omega(\mathrm{v}_1, w_1) + q\omega(\mathrm{v}_2, w_2),$$

where

$$\omega(\mathrm{v}_i, w_i) = \langle J\mathrm{v}_i, w_i \rangle \,.$$

From this formula one easily deduces

Theorem 3.2. *There are two kinds of Lagrangian subspaces of $\mathbb{R}^4$:*

    1. *Subspaces of the form*

(3.4) $$V_1 \oplus V_2 \subseteq \mathbb{R}^2 \oplus \mathbb{R}^2$$

        where each V_i is a one-dimensional subspace of $\mathbb{R}^2$.
    2. *Graphs of maps $A : \mathbb{R}^2 \to \mathbb{R}^2$ satisfying*

(3.5) $$A^*\omega = -\frac{p}{q}\omega \,.$$

Note that if $V \subseteq \mathbb{R}^4$ is a Lagrangian subspace of type 1, $V \cap S^3$ is a horizontal geodesic whose projection onto X goes through the north and south poles and if V is a Lagrangian subspace of type 2, $V \cap S^3$ is a horizontal geodesic whose projection onto X doesn't go through *either* the north or south pole. We'll call Lagrangians of type 1 *polar* Lagrangians and those of type 2 *non-polar* Lagrangians.

3.2. **The action of S^1 on the Grassmannian of two-dimensional subspaces of $\mathbb{R}^4$.** To simplify slightly the exposition below, we'll assume that p and q are odd. The linear action of S^1 on $\mathbb{R}^2 \oplus \mathbb{R}^2$, defined by

$$(3.6) \qquad \tau(e^{it}) = (\exp tpJ, \, \exp tqJ),$$

induces an action $\tau^{\#}$ of S^1 on the Grassmannian of two-dimensional subspaces of $\mathbb{R}^4$. If p and q are both odd this action isn't effective since $\tau(e^{i\pi}) = -I$. However we do get an effective action of $S^1/\{\pm 1\}$. Next note the following:

Lemma. *If V is a two-dimensional subspace of $\mathbb{R}^2 \oplus \mathbb{R}^2$ then either*

$$(3.7) \qquad V = V_1 \oplus V_2$$

where V_1 and V_2 are one-dimensional subspaces of $\mathbb{R}^2$, or

$$(3.8) \qquad V = \mathrm{graph}\, A$$

where $A : \mathbb{R}^2 \to \mathbb{R}^2$ is a map of the first summand of $\mathbb{R}^2 \oplus \mathbb{R}^2$ into the second, or

$$(3.8') \qquad V = \mathrm{graph}\, A'$$

where A' is a map of the second summand of $\mathbb{R}^2$ into the first.

It's easy to see that $S^1/\{\pm 1\}$ acts freely at points (3.7) of $G_2(\mathbb{R}^4)$. What about points of type (3.8) or (3.8')? These two cases are basically the same so let's concentrate on (3.8). Note that $\tau(e^{it})$ maps the subspace (3.8) into itself iff

$$(3.9) \qquad (\exp tqJ)A = A(\exp tpJ),$$

or, equivalently, iff

$$(3.10) \qquad (\cos qt)A = (\cos pt)A$$

and

$$(3.11) \qquad (\sin qt)JA = (\sin pt)AJ.$$

Hence if we exclude the trivial case $A = 0$ the element e^{it}, $0 < t < \pi$, of S^1 acts trivially on the Grassmannian at the point (3.8) iff

$$(3.12) \qquad\qquad JA = AJ \text{ and } e^{itp} = e^{itq}$$

or

$$(3.13) \qquad\qquad JA = -AJ \text{ and } e^{itp} = e^{-itq}.$$

But $e^{itp} = e^{itq}$ iff $e^{it(q-p)} = 1$ and hence iff

$$(3.14) \qquad\qquad t = \frac{\pi k}{q - p} \text{ for some } 0 < k < q - p,$$

and $e^{itp} = e^{iqt}$ iff $e^{it(p+q)} = 1$ and hence iff

$$(3.15) \qquad\qquad t = \frac{\pi k}{p + q} \text{ for some } 0 < k < q + p.$$

Moreover if (3.12) or (3.13) holds and $A \neq 0$ then A is invertible, and therefore graph A is also the graph of a map, A', from the second summand of $\mathbb{R}^2 \oplus \mathbb{R}^2$ to the first. Thus we've proved

Theorem 3.3. *Suppose p and q are odd. Then from the action $\tau^\#$ one gets an effective action of $S^1/\{\pm 1\}$ which fixes the subspaces $\mathbb{R}^2 \oplus \{0\}$ and $\{0\} \oplus \mathbb{R}^2$ and is elsewhere locally free. Moreover $\tau^\#$ is free except at subspaces V of type (3.8) with $AJ = \pm JA$. If $AJ = JA$ the isotropy group of V in S^1 is $\mathbb{Z}_{q-p}$, and if $AJ = -JA$ its isotropy group is $\mathbb{Z}_{p+q}$.*

If either p or q is even this result has to be slightly modified.

Theorem 3.4. *Suppose that either p or q is even. Then the action $\tau^\#$ fixes the subspaces $\mathbb{R}^2 \oplus \{0\}$ and $\{0\} \oplus \mathbb{R}^2$ and is elsewhere locally free except at subspaces of type (3.7) and at subspaces of type (3.8), with $AJ = \pm JA$. If V is of type (3.7) its isotropy group is $\mathbb{Z}_2$, and if V is of type (3.8) with $AJ = \pm JA$ its isotropy group is the same as in the theorem above.*

Since the action τ of S^1 on $\mathbb{R}^4$ preserves the symplectic form $\tilde{\omega}$, the induced action $\tau^\#$ on the Grassmannian of two-dimensional subspaces of $\mathbb{R}^4$ preserves the set of Lagrangian subspaces. Moreover, if $V = $ graph A is Lagrangian then by (3.5) $\det A = -\frac{p}{q}$. Therefore, if A is such that $AJ = JA$, graph A is *not* Lagrangian since $\det A$ is positive. Hence from the theorems above we get as a corollary:

Theorem 3.5. *If p and q are odd the action of $S^1/\{\pm 1\}$ on the Lagrangian Grassmannian is free except on the one-dimensional set*

$$(3.16) \qquad\qquad \{\text{graph } A_\lambda, \quad |\lambda|^2 = \frac{p}{q}\},$$

where $A_\lambda : \mathbb{C} \to \mathbb{C}$ is the map $A_\lambda z = \lambda \overline{z}$. If either p or q is even the action of S^1 is free except on the set of polar subspaces (3.7) (which are stabilized by $\mathbb{Z}_2$) and on the set (3.16). Moreover, no matter what the parity of p and q, the stabilizer group of points in the set (3.16) is $\mathbb{Z}_{p+q}$.

3.3. Geodesics on the weighted projective line.

By (3.6), $\tau^\#(e^{it})$ maps the subspace $V = \operatorname{graph} A_\lambda$ to the subspace $V' = \operatorname{graph} A_{\lambda'}$, where $\lambda' = e^{it(p+q)}\lambda$. Thus in particular S^1 acts transitively on the set of horizontal geodesics

$$(3.17) \qquad \left\{ \operatorname{graph} A_\lambda \cap S^3 , \quad |\lambda|^2 = \frac{p}{q} \right\}.$$

Therefore these horizontal geodesics project to a single geodesic in X. We will call this the *exceptional* geodesic. Note that this geodesic is non-polar, i.e., lies entirely in the non-singular part of X. Another key observation: Suppose that for some $0 < \theta < 2\pi$, $\tau(e^{i\theta})$ maps a Lagrangian subspace V onto itself. Then for every point v on the geodesic $V \cap S^3$, v and its image, $\tau(e^{i\theta})$v, project onto the same point in X. Now let $w \in V \cap S^3$ be a vector in V perpendicular to v and let

$$\gamma(t) = \cos tv + \sin tw \quad 0 \le t \le 2\pi$$

be the geodesic parametrization of the circle $V \cap S^3$, with $\gamma(0) = $ v, $\frac{d}{dt}\gamma(0) = w$ and $\gamma(t_0) = \tau(e^{i\theta}$v). Since the action of S^1 on the space of horizontal geodesics is locally free, $\theta = \frac{2\pi}{k}$ for some $k \in \mathbb{Z}_+$ and γ is the union of k geodesic segments

$$\gamma_m(t) = \tau(e^{\frac{i2\pi m}{k}})\gamma(t), \quad 0 < t < t_0.$$

Thus $t_0 = \theta$ and each of these segments projects onto a closed geodesic in X of length $\frac{2\pi}{k}$. Hence from Theorem 3.5 we conclude:

Theorem 3.6. *If p and q are odd the geodesics on the Zoll orbifold X are all of length π except for the exceptional geodesic which is of length $\frac{2\pi}{p+q}$. If either p or q is even the geodesics on X are all of length 2π except for the polar geodesics (which are of length π) and the exceptional geodesic (which is again of length $\frac{2\pi}{p+q}$).*

Thus, coming back to the results of §1 we've proved

Theorem 3.7. *From the spectrum of the Laplace operator on X one can "hear" whether p or q is odd and can also hear their sum $p + q$.*

4. Circle actions on $\mathbb{C}^d$

As in the previous sections we'll identify $\mathbb{C}^d$ with $\mathbb{R}^{2d}$, i.e., with the sum

$$(4.1) \qquad \mathbb{R}^2 \oplus \cdots \oplus \mathbb{R}^2$$

of d copies of $\mathbb{R}^2$, and we'll view the standard representation of S^1 on $\mathbb{C}^d$ with weights N_i, $i = 1, \ldots, d$ as an $\mathbb{R}$-linear representation:

$$(4.2) \qquad \tau(e^{it}) = (\exp N_1 tJ, \ldots, \exp N_d tJ).$$

From (4.2) we get an action, $\tau^\#$, of S^1 on the Grassmannian of two-dimensional subspaces of $\mathbb{R}^{2d}$, and we'll try to ascertain below what the stabilizer groups are for this action. Our approach to this problem will be an inductive one: Assuming we know what these isotropy groups are for $\mathbb{R}^{2d-2}$ we'll try to determine what they are for $\mathbb{R}^{2d}$. Henceforth we'll assume that the N_i's *are greater than one and are pair-wise relatively prime.*

We first note that each of the summands of (4.1) is a fixed point for the action $\tau^\#$, and that on the complement of these d fixed points this action is locally free. If all the N_i's are odd then $\tau(e^{i\pi}) = -I$ and therefore $\tau^\#(e^{i\pi})$ is the identity map, and so, as in §4, one gets from $\tau^\#$ an effective action of $S^1/\{\pm 1\}$. If, one the other hand, N_1 is even, the action $\tau^\#$ itself is an effective action; however, if V is a two-dimensional subspace of the sum

$$(4.3) \qquad \{0\} \oplus \mathbb{R}^2 \oplus \cdots \oplus \mathbb{R}^2$$

its stabilizer contains $\mathbb{Z}_2$. Also, no matter what the parity of the N_i's is, if V is a subspace which is spanned by v_1 and v_2 where v_1 lies in (4.3) and v_2 lies in the first summand of (4.1) then its stabilizer is *exactly* $\mathbb{Z}_2$.

Definition. We will call a two-dimensional subspace, V, of $\mathbb{R}^{2d}$ *exceptional* if its stabilizer is not the identity and is not $\mathbb{Z}_2$.

We'll now outline an inductive method for determining these exceptional points in $\mathrm{Gr}_2(\mathbb{R}^{2d})$ and their stabilizers. Let

$$(4.4) \qquad \mathbb{R}^2 \oplus \{0\} \oplus \cdots \oplus \{0\}$$

be the first summand of (4.1).

Lemma 4.1. *If V is exceptional then its projection onto (4.4) is either zero or two-dimensional.*

Proof. If the projection is one-dimensional then V is spanned by vectors v_1 and v_2 where v_1 lies in (4.4) and v_2 lies in (4.3). Therefore, as we noted above, the stabilizer of V is $\mathbb{Z}_2$ and V is not exceptional.

$\square$

The same argument shows

Lemma 4.2. *If V is exceptional then its projection onto (4.3) is either zero or two-dimensional.*

From these results we conclude

Lemma 4.3. *If V is exceptional then either it has to be the space (4.4), or be contained in the space (4.3), or be the graph of a map A of (4.4) into (4.3). Moreover this map has to be injective.*

If V is the space (4.4) then it is a fixed point for the action $\tau^{\#}$, and if V is contained in (4.3) then it is an exceptional subspace of $\mathbb{R}^{2d-2}$ and hence determined by our induction procedure. Let's focus therefore on the case where $V = \operatorname{graph} A$, A an injective map of (4.4) into (4.3). We can write this map as a direct sum

$$A = (A_2, \ldots, A_d),$$

where A_k is a map of $\mathbb{R}^2$ into the kth summand of (4.3). The following result is a consequence of (3.12) and (3.13):

Lemma 4.4. *Suppose V is stabilized by $\tau^{\#}(e^{it})$ with $0 < t < 2\pi$ and $t \neq \pi$. Then, for each $k \in \{2, \ldots, d\}$, either*

$$\begin{aligned}
\text{(i)} \quad & A_k = 0 \text{ or} \\
\text{(4.5)} \qquad \text{(ii)} \quad & JA_k = A_k J \text{ and } e^{itN_1} = e^{itN_k} \text{ or} \\
\text{(iii)} \quad & JA_k = -A_k J \text{ and } e^{itN_1} = e^{-itN_k}.
\end{aligned}$$

Some notation: Let $S_1(A)$ be the set of k's for which (ii) holds and let $S_2(A)$ be the set of k's for which (iii) holds. Also, if $k \in S_1(A)$ let $\sigma_k = |N_k - N_1|$ and if $k \in S_2(A)$ let $\sigma_k(A) = N_1 + N_k$.

Putting together the lemmas above we get the main result of this section.

Theorem 4.5. *Let $A = (A_2, \ldots, A_d)$ be a above, and let $S(A) = S_1(A) \cup S_2(A)$. Then the graph of A is exceptional if the components of A satisfy (4.5) and the greatest common divisor of the set of numbers*

$$\text{(4.6)} \qquad\qquad \{\sigma_k(A), \quad k \in S(A)\}$$

is greater than 2.

Thus one can determine the isotropy groups of the exceptional points of $\operatorname{Gr}_2(\mathbb{R}^{2d})$ by the following procedure:

1. Fix two disjoint subsets S_1 and S_2 of the set $\{2, \ldots, d\}$.
2. For each $k \in S_1$ let $\sigma_k = |N_k - N_1|$.

3. For each $k \in S_2$ let $\sigma_k = N_k + N_1$.
4. Let $\sigma(S_1, S_2)$ be the greatest common divisor of the set of numbers
$$\{\, \sigma_k, \quad k \in S_1 \cup S_2 \,\}.$$

Then if $N = \sigma(S_1, S_2)$ is greater than 2, $\mathbb{Z}_N$ is the isotropy group of an exceptional point, $V = \operatorname{graph} A$, of $\operatorname{Gr}_2(\mathbb{R}^{2d})$.

This observation gives us, by induction, the result below.

Theorem 4.6. *For $1 \le r \le d-1$ let S_1 and S_2 be disjoint subsets of the set $\{r+1, \ldots, d\}$, and let $\sigma_k = |N_k - N_r|$ if k is in S_1 and $\sigma_k = N_k + N_r$ if k is in S_2. Then, if the greatest common divisor $N = \sigma(S_1, S_2)$ of the set of numbers $\{\, \sigma_k,\ k \in S_1 \cup S_2 \,\}$ is greater than 2, $\mathbb{Z}_N$ is the isotropy group of an exceptional point of $\operatorname{Gr}_2(\mathbb{R}^d)$.*

5. Geodesics on weighted $(d-1)$-dimensional projective spaces

As mentioned in the introduction, this space is the quotient of S^{2d-1} by the circle action (4.2), i.e., it is the space

$$(5.1) \qquad\qquad X = S^{2d-1}/S^1.$$

(The title of this section is a bit of a misnomer since X, as a real C^∞ Riemannian orbifold, is $(2d-2)$-dimensional.) There are $\binom{d}{2}$ totally-geodesic embedded weighted projective lines $X_{i,j}$ $1 \le i < j \le d$ in X, corresponding to the direct sum of two complex coordinate lines in $\mathbb{C}^d$. As we showed in §3, each weighted projective line has a unique exceptional geodesic $\gamma_{i,j}$, and it has length

$$\ell_{i,j} = \frac{2\pi}{N_i + N_j}$$

(assuming, as we always do in this paper, that the weights are pairwise relatively prime). In this section we investigate conditions on the weights which guarantee that the lengths $\ell_{i,j}$, $1 \le i < j \le d$, are the *shortest* in the length spectrum of X. As a corollary of the discussion of §2 in such cases one can "hear" the multiset of the $\ell_{i,j}$, and therefore the multiset of pair-wise sums $\{N_i + N_j, 1 \le i < j \le d\}$. This analysis requires that we have a description of the lengths of the exceptional geodesics on X (other than the $\gamma_{i,j}$).

Let $\tilde\omega \in \Lambda^2(\mathbb{R}^{2d})^*$ be the symplectic form

$$\tilde\omega(\mathrm{v}, \mathrm{w}) = \sum_{i=1}^{d} N_i \langle J\mathrm{v}_i, \mathrm{w}_i \rangle$$

where v_i and w_i are the projections of v and w onto the ith summand of (4.1). The geodesics on the sphere S^{2d-1} are just the intersections $V \cap S^{2d-1}$, where V is a two-dimensional subspace of $\mathbb{R}^{2d}$, and one can show by exactly the same argument as in §3 that $V \cap S^{2d-1}$ is *horizontal* if and only if $\iota_V^* \tilde{\omega} = 0$, i.e., if and only if V is an isotropic two-dimensional subspace of $\mathbb{R}^{2d}$. Given such a geodesic, the projection $S^{2d-1} \to X$ maps $V \cap S^{2d-1}$ onto a closed geodesic of X of period $\frac{2\pi}{m}$, where $\mathbb{Z}_m$ is the stabilizer of V with respect to the action $\tau^\#$ of S^1 on $\mathrm{Gr}_2(\mathbb{R}^{2d})$. In particular, suppose $V = \mathrm{graph}(A)$ as in the previous section. Then V is isotropic if and only if

$$(5.2) \qquad N_1 + \sum_{m=2}^{d} N_m \det A_m = 0,$$

and if this is the case there exists m such that $\det A_m$ is negative. Therefore, if V is exceptional and isotropic, (4.5)(iii) must hold for some m, or, equivalently, $S_2(A) \neq \emptyset$. It follows that the stabilizer group of V has to be contained in $\mathbb{Z}_{N_1+N_m}$ for some m.

We summarize:

Proposition 5.1. *The lengths of the exceptional geodesics on X are of the form $\ell_k = 2\pi/k$, where k is the greatest common divisor of a set of numbers of the form*

$$\mathcal{F}_{S_1, S_2} = \{ \sigma_m \; ; \; m \in S_1 \cup S_2 \},$$

and where S_1, S_2 are sets such that

$$S_1 \cup S_2 \subset \{r+1, \ldots d\}, \;\; S_1 \cap S_2 = \emptyset, \;\; S_2 \neq \emptyset$$

and such that $g.c.d.(\mathcal{F}_{S_1, S_2}) > 2$. Here

$$\sigma_m = \{ \begin{array}{ll} |N_m - N_r| & \text{if } m \in S_1 \\ N_m + N_r & \text{if } m \in S_2. \end{array}$$

Note that if one takes $S_1 = \emptyset$ and $S_2 = \{m\}$ (a singleton), then $k = N_r + N_m$ gives rise to the length of $\gamma_{r,m}$.

In low dimensions we have the following results:

Theorem 5.2. *Suppose all the N_i's are pair-wise relatively prime. Then:*
(1) If $d = 3$, one can "hear" the weights N_1, N_2 and N_3.
(2) If $d = 4$, one can "hear" all the pairwise sums $\{ N_i + N_j, \; i < j \}$, which will determine at most two different choices of N_i's.

5.1. **The case** $d = 3$. The exceptional geodesics other than the $\gamma_{i,j}$, if they exist, have to be associated with isotropics of the form $V = \operatorname{graph} A$, where $A = (A_2, A_3)$ and each of the A_i's is bijective. If we order the N_i's so that $N_1 < N_2 < N_3$, these geodesics will have lengths of the form $2\pi/k$ where the possibilities for k are

$$(5.3) \qquad\qquad k \;=\; \operatorname{g.c.d}(N_1 + N_3, N_2 - N_1)$$

or

$$(5.4) \qquad\qquad k \;=\; \operatorname{g.c.d}(N_1 + N_2, N_3 - N_1)$$

or

$$(5.5) \qquad\qquad k \;=\; \operatorname{g.c.d}(N_1 + N_2, N_1 + N_3) \,.$$

(These possibilities correspond to the set S_2 of Theorem 4.5 being equal to $\{3\}$, $\{2\}$ and $\{2,3\}$, respectively.) If k is given by (5.3) then $k < N_2 < N_1 + N_2$, and the geodesic corresponding to k is strictly longer than the geodesics γ_1, γ_2 and γ_3. If k is given by (5.4) then $k < N_1 + N_3$ and therefore the geodesic is strictly longer than γ_2 and γ_3. The geodesic will also be longer than γ_1 unless $N_1 + N_2$ divides $N_3 - N_1$, or, equivalently (since $N_2 + N_3 = N_3 - N_1 + N_1 + N_2$), unless $N_1 + N_2$ divides $N_2 + N_3$. Similarly, if k is given by (5.5) the geodesic will be strictly longer unless $N_1 + N_2$ divides $N_1 + N_3$. Thus if neither of these two worst case scenarios occurs, γ_1, γ_2 and γ_3 are the three shortest geodesics on X. Moreover, they lie on the non-singular part of X and are isolated and non-degenerate. Hence by [DG], γ_i makes a contribution of the form

$$(5.6) \qquad\qquad\qquad c_i(t - T_i)_+^{-1} + \cdots$$

to the wave trace with $c_i \neq 0$ where T_i is the period of γ_i and hence one can "hear" the T_i's, i.e., one can hear $N_1 + N_2$, $N_1 + N_3$ and $N_2 + N_3$ and hence one can hear N_1, N_2 and N_3.

Let's next examine the first worst case scenario: $N_1 + N_2$ divides $N_1 + N_3$. Then $V = \operatorname{graph} A$ is isotropic and stabilized by $\mathbb{Z}_{N_1+N_2}$ if and only if $A = (A_{\lambda_2}, A_{\lambda_3})$ with

$$(5.7) \qquad\qquad N_1 - N_2|\lambda_2|^2 - N_3|\lambda_3|^2 = 0$$

(see (3.16)). Thus the set of geodesics stabilized by $\mathbb{Z}_{N_1+N_2}$ consists of the isolated geodesic γ_1 on the projective line X_1, and completely disjoint from it the 2-parameter family of geodesics lying on the quotient by S^1 of the three sphere (5.7). In this case the wave trace contains contributions at $T_1 = 2\pi/(N_1 + N_2)$ from both γ_1 and from the geodesics

(5.7). However, since the geodesics (5.7) also lie on the non-singular part of X the clean intersection techniques of [DG] show that their contribution to the wave trace is a singularity of the form

$$(5.8) \qquad\qquad c(t - T_1)^{-3} + \cdots$$

which dominates the singularity (5.6), so again one can hear $N_1 + N_2$.

A similar situation occurs if $N_1 + N_2$ divides $N_2 + N_3$. In this case one gets again a 2-parameter family of geodesics on X corresponding to A's of the form $A = (A_{\lambda_2}, A_{\lambda_3})$ where

$$(5.9) \qquad\qquad N_2|\lambda_2|^2 = N_1 + N_3|\lambda_3|^2 \,.$$

The projection of this set onto the space of geodesics on X isn't closed, but its closure consists of the geodesics in this set plus the geodesic γ_3 (which is stabilized by $\mathbb{Z}_{N_2+N_3}$ and hence by $\mathbb{Z}_{N_1+N_2}$). Thus the clean intersection techniques of [DG] show that the contribution of this family to the wave trace is of the form (5.8), which again dominates the contribution (5.6) coming from γ_1. Thus, in both these two worst cases scenarios one can hear $N_1 + N_2$ and therefore all three N_i.

5.2. **The case $d = 4$.** Without loss of generality, we suppose $N_1 < N_2 < N_3 < N_4$. There are now $\binom{4}{2} = 6$ embedded totally geodesic weighted projective lines, each one with a unique exceptional geodesic $\gamma_{i,j}$, of length $\frac{2\pi}{N_i+N_j}$ (these are the "desirable" lengths that we wish to "hear"). Note that $\gamma_{4,3}$ is the shortest geodesic, $\gamma_{4,2}$ the second shortest, $\gamma_{2,1}$ the longest and $\gamma_{3,1}$ the second longest of these geodesics, and in general we can't order the lengths of $\gamma_{4,1}$ with $\gamma_{3,2}$ (in fact they could be equal, e.g.: $17 + 3 = 13 + 7$).

The lengths of undesirable exceptional geodesics are of the form $L_k = \frac{2\pi}{k}$, k a positive integer. We want to bound the possible values of L_k from below, to determine which desirable lengths are spectrally determined.

According to our inductive procedure for finding the possible k's, we should first consider possible two-dimensional planes contained in

$$\{0\} \oplus \mathbb{R}^2 \oplus \mathbb{R}^2 \oplus \mathbb{R}^2.$$

These correspond to the exceptional geodesics of a $d = 3$ weighted projective space, with weights N_2, N_3, N_4. As we have seen, the possibilities for k are the greatest common divisor of the following sets:
(5.10)
$$\{N_4 + N_2, \, N_3 - N_2\}, \quad \{N_3 + N_2, \, N_4 - N_2\}, \quad \{N_3 + N_2, \, N_4 + N_2\}.$$

The remaining exceptional geodesics are graphs of maps $A = (A_2, A_3, A_4)$. The possible values of k are the greatest common divisors of the following seven sets:

(5.11)
$$\{N_2 + N_1, N_3 + N_1\}, \quad \{N_2 + N_1, N_3 + N_1, N_4 - N_1\},$$
$$\{N_2 + N_1, N_4 + N_1\}, \quad \{N_2 + N_1, N_4 + N_1, N_3 - N_1\},$$
$$\{N_3 + N_1, N_4 + N_1\}, \quad \{N_3 + N_1, N_4 + N_1, N_2 - N_1\}$$

and

(5.12)
$$\{N_2 + N_1, N_3 + N_1, N_4 + N_1\}.$$

Using only that the g.c.d. of any set of numbers is less than or equal to any element in the set, one can check that in all cases $k \leq N_3 + N_2$, so we can always hear the lengths of the two shortest $\gamma_{i,j}$ and determine $N_4 + N_3$ and $N_4 + N_2$. In addition, we claim that we can always hear $N_4 + N_1$, because the g.c.d.'s of all the sets listed above are all less than $N_4 + N_1$. The only set for which this is not immediately obvious is $\{N_3 + N_2, N_4 + N_2\}$; however, note that

$$\text{g.c.d}\{N_3 + N_2, N_4 + N_2\} = \text{g.c.d}\{N_3 + N_2, N_4 - N_3\} < N_4 < N_4 + N_1.$$

One can also check that the g.c.d of the sets (5.11) satisfies $k \leq N_3 + N_1$, and the g.c.d of the sets (5.12) $k \leq N_1 + N_2$. Therefore the lengths of the corresponding geodesics are greater than the length of $\gamma_{3,2}$. If the g.c.d.'s of the sets (5.10) is less than $N_3 + N_2$, then all undesirable exceptional geodesics are longer than $\gamma_{3,2}$ as well. However, by our analysis of the "worst case scenario" when one of the g.c.d.'s of the sets (5.10) equals $N_3 + N_2$, we can conclude that we can always hear $N_3 + N_2$.

In conclusion, the multiset $\{N_2 + N_3, N_1 + N_4, N_2 + N_4, N_3 + N_4\}$ is always spectrally determined. In this set we can identify the individual sums $N_3 + N_4$ and $N_2 + N_4$ as the largest and second largest elements, respectively. Moreover, although we are not able to distinguish between the two remaining elements in the set which one is $N_2 + N_3$ and which one is $N_1 + N_4$, we can form their sum,

$$N_1 + N_2 + N_3 + N_4,$$

and from this determine $N_1 + N_2$ and $N_1 + N_3$, (by subtracting the known quantities $N_3 + N_4$ and $N_2 + N_4$). This shows that we can always hear the set of all six sums of pairs $\{N_i + N_j, i < j\}$.

To reconstruct the possible weights giving rise to a given multiset of sums of pairs, proceed as follows: $N_1 + N_2$ and $N_1 + N_3$ are the smallest and second smallest elements of the set, respectively. However, $N_2 + N_3$ may be the third or fourth element in the set if the set has six elements (the set may have only five elements, exactly when $N_2 + N_3 = N_4 + N_1$,

in which case this sum is the middle element and the reconstruction of the weights is unique). Each one of those possibilities for $N_2 + N_3$ gives a unique solution for N_1, N_2, N_3, and the remaining elements in the set uniquely determine the fourth weight.

It is clear that picking $N_2 + N_3$ to be the third or the fourth element from a set of six distinct pairwise sums gives different solutions $\{N_j\}$. However, the two solutions "usually" do not both consist of four pairwise relatively prime numbers, although this can happen. For example,

$$\{25, 29, 41, 61\} \text{ and } \{17, 37, 49, 53\}$$

have the same set of pairwise sums.

5.3. **Remarks for d general.**
In general, if the $\binom{d}{2}$ shortest geodesics of X are the $\gamma_{i,j}$ then one can "hear" the weights if d is not a power of two. A way to ensure that the $\gamma_{i,j}$ are the $\binom{d}{2}$ shortest geodesics is the following:

Theorem 5.3. *If $N_1 < N_2 < \ldots < N_d$, $N_d \leq 2N_1$ and the list $N_i + N_j$, $1 \leq i < j \leq d$, contains no repeated elements, then the $\gamma_{i,j}$ are the $\binom{d}{2}$ shortest geodesics, (and therefore, by the result in the appendix, the set of weights is spectrally determined if d is not a power of two).*

Proof. Let $\frac{2\pi}{k}$ be an undesirable length of an exceptional geodesic. How short can the exceptional geodesic be or, equivalently, how large can k be? Using Proposition 5.1, it is not hard to see that

$$k \leq \text{g.c.d}\{N_i + N_j,\ N_l + N_m\}$$

for some $i < j$, $l < m$ and $(i, j) \neq (l, m)$. Thus it suffices to show that such g.c.d.'s are less than $N_1 + N_2$.

First, the condition $N_d \leq 2N_1$ implies that $N_l + N_m < 2(N_i + N_j)$. Therefore

$$\text{g.c.d}\{N_i + N_j,\ N_l + N_m\} \neq N_i + N_j$$

(since $a = \text{g.c.d}\{a, b\} \wedge b < 2a \Rightarrow b = a$, and we assumed that the list $N_i + N_j$, $1 \leq i < j \leq d$, contains no repeated elements). It follows that $k \leq (N_i + N_j)/2 < N_1 + N_2$. $\qquad\square$

APPENDIX A.

We present in this appendix a proof of the following:

Theorem A.1. *If d is not a power of two, a set of d real numbers $\{N_1, \ldots, N_d\}$ is determined by the multiset of sums of pairs $\{N_j + N_k : j < k\}$.*

Proof. The proof is based on the identity

$$\sum_{1 \le i < j \le d} (N_i + N_j)^k = (d - 2^k) \sum_{i=1}^{d} N_i^k + \frac{1}{2} \sum_{m=1}^{k-1} \binom{k}{m} (\sum_{i=1}^{d} N_i^{k-m})(\sum_{i=1}^{d} N_i^m).$$

This shows that if d is not a power of two then one can find $\sum N_i^k$ from $\sum_{1 \le i < j \le d}(N_i + N_j)^k$ and the sums $\sum N_i^m$ with $m < k$. Proceeding inductively (starting from $\sum_{1 \le i < j \le d}(N_i + N_j) = (d - 1) \sum N_i$), we can conclude that the elementary symmetric functions on the sums of pairs determine the elementary symmetric functions on the N_j. $\square$

Finally, we remark that one can generalize the previous discussion of the $d - 2$ case to obtain an algorithm for finding all possible sets of N_i's, given the multiset of their pair-wise sums. This algorithm shows that, if d is a power of 2 and $d > 2$, then the multiset $\{N_i + N_j\}$ will determine at most $d - 2$ different choices of N_i's. However, it seems very hard to determine what the sharp bound of possible choices of N_i's is, for a given d.

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